SPECTRAL VACUUM MECHANISM — PART XLVIII

Algebraic Vacuum, Spectral Constraints, and Renormalization Structure

M. Nemirovsky

ORCID: 0009-0005-2572-5301

2026

Abstract

We present a fully consistent operator-based formulation of the Spectral Vacuum Mechanism and establish a rigorous bridge from algebraic vacuum structure to effective spectral geometry and renormalization behaviour.

Starting from the spectral action

$$\mathbf{S[H] = -\alpha\, Tr(H^2) + \beta\, Tr(H^4),}$$

we derive the stationary vacuum equation

$$\mathbf{H_0^3 - \mu^2 H_0 = 0, \quad \mu^2 = \alpha/(2\beta),}$$

which constrains the vacuum spectrum to $\{0, \pm\mu\}$. We compute the second variation explicitly and obtain the full block-diagonal Hessian structure.

We prove two fundamental structural theorems. First, the Finite-Spectrum No-Go Theorem (Theorem 6.1): no finite-spectrum operator can reproduce ultraviolet heat-kernel scaling of the form

$$\mathbf{P(t) \sim t^{-d/2}, \quad d > 0.}$$

Second, the Projection Stability Theorem (Theorem 7.1–7.2): spectral finiteness is preserved under orthogonal projections and algebraic operations, implying the impossibility of generating RG scaling within the algebraic vacuum sector.

These results imply that non-trivial ultraviolet behaviour cannot emerge at the exact algebraic level. We therefore formulate the Effective Realization Principle and construct an explicit admissible class of effective operators K_eff compatible with the vacuum constraints but possessing infinite spectral support.

For this class we derive the heat-kernel expansion

$$\mathbf{P(t) = a_0\, t^{-2} + a_2\, t^{-1} + a_4 + O(t),}$$

and establish a one-to-one correspondence between the Seeley–DeWitt coefficients (a_0, a_2, a_4) and internal operator structure.

We define the ultraviolet spectral dimension d_UV as the minimiser of the reconstruction functional $\Psi(d)$ and show that it coincides with the fixed point of the spectral renormalization group flow:

$$\mathbf{\beta\_spec(d\_UV) = 0.}$$

The results are organised into three strictly separated levels: (i) exact algebraic vacuum structure; (ii) universal structural obstructions for all finite-spectrum operators; and (iii) effective spectral geometry arising from admissible operators. This establishes the Spectral

Vacuum Mechanism as a closed framework in which vacuum algebra, spectral geometry, and renormalization flow are derived from a single operator principle.

Keywords: spectral action; vacuum operator; Hessian spectrum; heat-kernel coefficients; Seeley–DeWitt expansion; no-go theorem; effective spectral dimension; renormalization group; Spectral Vacuum Mechanism

Main Results

Summary of principal results

(MR1) Vacuum structure (exact). The stationary vacuum H_0 of $S[H] = -\alpha\, Tr(H^2) + \beta\, Tr(H^4)$ satisfies $H_0^3 - \mu^2 H_0 = 0$ and has spectrum confined to $\{0, \pm\mu\}$, $\mu^2 = \alpha/(2\beta)$. The Hessian $L_{H_0}^{Hess}$ has block-diagonal spectrum $\{0, \mu^2, 2\mu^2, 3\mu^2\}$ and the algebraic sector operator $L_{H_0}^{alg}$ has spectrum $\subset \{0, \mu^2, 4\mu^2\}$. These are exact algebraic results requiring no approximation.

(MR2) Finite-Spectrum No-Go (quantitative theorem). Theorem 6.1: for any finite-spectrum operator and any window W, $\inf_q \sup_{t\in W}|t^{d/2}P(t) - \text{poly}| \geq c(d,W) > 0$, hence $\Psi(d) > 0$ for all $d > 0$. Corollary 6.2: $d_{eff} \to 0$ (not a non-trivial RG fixed point). Corollary 6.3: H_0, L^{Hess}, L^{alg} cannot produce UV scaling, spectral dimension, or RG flow.

(MR3) Projection Stability and algebraic closure (RG-enhanced). Theorem 7.1: $\Psi_{A_\Pi}(d) > 0$ for all projections Π. Theorem 7.2: the class of finite-spectrum operators is closed under projections, commutators, and polynomials, and $\Psi > 0$ throughout. Corollary 7.3: no algebraic sequence from H_0 can produce $\beta_{spec} = 0$ at finite $d > 0$. Theorem 8.4: UV scaling $P(t) \sim t^{-d/2}$ requires infinite spectral support.

(MR4) Effective Realization Principle and admissible construction. The vacuum defines an admissible class via conditions (A1)–(A5). Proposition 8.2 proves a canonical block-diagonal decomposition $K_{eff} = K_+ \oplus K_- \oplus K_0$ with vacuum-core consistency. Theorem 8.3 proves existence: for every integer $m \geq 1$ and every multiplicity triple (n_+, n_-, n_0), there exists an admissible K_{eff} with $P(t) \sim C\, t^{-m/2}$; the construction uses the Laplace–Beltrami operator on a closed m-dimensional Riemannian manifold tensored over each sector.

(MR5) Heat-kernel structure and St46 link. For admissible K_{eff}, P(t) admits the SDW expansion (9.2). The canonical susceptibility satisfies $\kappa_{can} = a_2^2/a_4$ (Proposition 9.1 via $\chi_{eff} = a_2^2/(a_0\, a_4)$), connecting the vacuum sector data to the spectral reconstruction invariant of St46.

(MR6) Renormalization fixed point. $\Psi(d)$ is continuous and attains its minimum d_{UV} (Theorem 9.2). If $\varepsilon_{rel}(d^*, W_n) \to 0$ along UV windows, then $d_{eff} \to d^*$ and $\beta_{spec} \to 0$ (Theorem 9.3). The effective spectral dimension is a UV fixed point of the spectral RG flow.

Contents

1. Introduction

The Spectral Vacuum Mechanism (SVM) aims to derive physical structure from an operator-based vacuum theory without imposing background geometry. The central object is a self-adjoint operator H governed by a spectral action that is unitarily invariant and spectrally defined. Parts St43–St47 developed the heat-kernel expansion, spectral dimension extraction, susceptibility invariants, and the transition-scale framework. The present work (St48) establishes the operator-theoretic foundation underlying all of these: it identifies the algebraic vacuum, proves structural no-go theorems that separate the vacuum layer from the geometric layer, and constructs the admissible class of effective operators on which the geometric analysis of the earlier parts is valid.

A key structural tension motivates the work. The stationary vacuum of the spectral action has a finite three-point spectrum $\{0, \pm\mu\}$. Yet physically meaningful spectral geometry requires heat-kernel asymptotics $P(t) \sim t^{-d/2}$ as $t \to 0$, which requires a continuously distributed or very rich discrete spectrum. This tension is resolved by establishing a two-layer structure:

(i) The algebraic vacuum layer: exact, finite, analytically rigid. H_0 and all operators algebraically derived from it have finite spectra and cannot support geometry (Theorems 6.1, 7.1).

(ii) The effective spectral layer: the admissible class of operators K_{eff} consistent with vacuum constraints. Spectral geometry, heat-kernel coefficients, and renormalization structure arise here.

The logical chain of the paper is:

vacuum H_0 → constraints → admissible K_{eff} → heat kernel → RG → geometry.

Convention.

H is self-adjoint on a finite-dimensional Hilbert space $\mathscr{H}$ of dimension N. Tr denotes the operator trace. $[A,B] = AB - BA$. $\alpha, \beta > 0$ throughout. $\mu^2 = \alpha/(2\beta)$.

2. Spectral Action

Let $\mathscr{H}$ be a finite-dimensional complex Hilbert space and $H \in \mathrm{End}(\mathscr{H})$ self-adjoint. The spectral action is

$$S[H] = -\alpha\, \mathrm{Tr}(H^2) + \beta\, \mathrm{Tr}(H^4). \tag{2.1}$$

Properties: (i) unitary invariance $S[UHU^*] = S[H]$ for all unitary U; (ii) spectral definability — S[H] depends only on eigenvalues; (iii) bounded below by $-N\alpha^2/(4\beta)$ since for each eigenvalue λ_i, the function $f(\lambda) = -\alpha\lambda^2 + \beta\lambda^4$ achieves minimum $-\alpha^2/(4\beta)$ at $\lambda^2 = \mu^2$; (iv) non-convex — the negative quadratic term creates a double-well structure essential for a non-trivial vacuum.

Proposition 2.1 (Spectral characterisation of stationary points). Any stationary point H_0 of the spectral action is spectrally supported on the critical set of $f(\lambda) = -\alpha\lambda^2 + \beta\lambda^4$, i.e.

$$\mathrm{spec}(H_0) \subset \{0, \pm\mu\}, \quad \mu = \sqrt{\alpha/(2\beta)}. \tag{2.2}$$

Proof. Stationarity reduces to $f'(\lambda) = 0$ with

$$f'(\lambda) = -2\alpha\lambda + 4\beta\lambda^3 = 0, \tag{2.3}$$

hence $\lambda(\lambda^2 - \mu^2) = 0$, giving $\lambda \in \{0, \pm\mu\}$. ∎

3. Vacuum Structure

3.1 Stationarity condition

Setting $\delta S/\delta H = -2\alpha H + 4\beta H^3 = 0$ at $H = H_0$ and factoring:

$$H_0(H_0^2 - \mu^2) = 0, \quad \mu^2 = \alpha/(2\beta). \tag{3.1}$$

By the spectral mapping theorem, eigenvalues of H_0 satisfy $\lambda(\lambda^2 - \mu^2) = 0$, giving

$$\mathrm{spec}(H_0) \subset \{0, +\mu, -\mu\}. \tag{3.2}$$

3.2 Projection decomposition

Let P_+, P_-, P_0 be the spectral projections onto the $+\mu$, $-\mu$, and 0 eigenspaces respectively, with $n_+ = \mathrm{rank}\, P_+$, $n_- = \mathrm{rank}\, P_-$, $n_0 = \mathrm{rank}\, P_0$, and $n_+ + n_- + n_0 = N$. Then

$$H_0 = \mu(P_+ - P_-), \quad P_+ + P_- + P_0 = I, \quad P_iP_j = \delta_{ij} P_i. \tag{3.3}$$

The vacuum is characterised by the triple (n_+, n_-, n_0) and the scale μ. It is algebraically rigid: given these data, H_0 is uniquely determined.

Proposition 3.1 (Vacuum classification and energy). Any stationary vacuum H_0 is uniquely determined up to unitary equivalence by the triple (n_+, n_-, n_0). Its action value is

$$S[H_0] = -(\alpha^2/4\beta) \cdot (n_+ + n_-). \tag{3.4}$$

Proof. Using $H_0^2 = \mu^2(P_+ + P_-)$ and $H_0^4 = \mu^4(P_+ + P_-)$:

$$\mathrm{Tr}(H_0^2) = \mu^2(n_+ + n_-), \quad \mathrm{Tr}(H_0^4) = \mu^4(n_+ + n_-). \tag{3.5}$$

Substituting into $S[H_0] = -\alpha\, \mathrm{Tr}(H_0^2) + \beta\, \mathrm{Tr}(H_0^4)$:

$$S[H_0] = (-\alpha\mu^2 + \beta\mu^4)(n_+ + n_-) = (-\alpha \cdot \alpha/(2\beta) + \beta \cdot \alpha^2/(4\beta^2))(n_+ + n_-)$$
$$= (-\alpha^2/(2\beta) + \alpha^2/(4\beta))(n_+ + n_-) = -(\alpha^2/4\beta)(n_+ + n_-).$$ ∎

Corollary. The zero sector (n_0) does not contribute to vacuum energy. Vacua with $n_0 = 0$ are energetically saturated; $n_0 > 0$ leads to instability directions (see Section 4).

Interpretation. The vacuum solution H_0 defines a rigid algebraic structure: a finite spectral decomposition into sectors (P_+, P_-, P_0) with fixed multiplicities (n_+, n_-, n_0) and scale μ. However, this structure does not by itself encode geometric information. In particular, as will be proved in Sections 6–7, any operator constructed purely at the algebraic level remains finite-spectrum and therefore cannot support ultraviolet scaling of the form P(t) ~ t^{−d/2}. This establishes the necessity of a second (effective) layer of description.

4. Second Variation and Hessian Spectrum

4.1 Second variation

A full derivation is given in Appendix A. The result is

$$\delta^2 S(K) = -\alpha\, \mathrm{Tr}(K^2) + 4\beta\, \mathrm{Tr}(H_0^2K^2) + 2\beta\, \mathrm{Tr}(H_0KH_0K). \quad (4.1)$$

Defining the Hessian operator $L_{H_0}^{Hess}(K) = H_0^2K + H_0KH_0 + KH_0^2$, this compacts to

$$\delta^2 S(K) = \mathrm{Tr}(K\,[-\alpha I + 2\beta\, L_{H_0}^{Hess}]\,K). \quad (4.2)$$

4.2 Block-diagonal structure

We decompose End($\mathcal{H}$) according to the eigenspace sectors of H_0. A perturbation K decomposes as $K = \Sigma_{a,b} K_{ab}$ where K_{ab} maps the b-eigenspace to the a-eigenspace ($a, b \in \{+, -, 0\}$). The Hessian acts diagonally on each block:

$$L_{H_0}^{Hess}(K_{ab}) = (\lambda_a^2 + \lambda_a \lambda_b + \lambda_b^2)\, K_{ab}, \quad (4.3)$$

where $\lambda_+ = \mu$, $\lambda_- = -\mu$, $\lambda_0 = 0$. Computing each block:

Block (a,b)	**$\lambda_a^2 + \lambda_a\lambda_b + \lambda_b^2$**	**$2\beta \times$ eigenvalue**	**Stability condition**
(+,+) or (−,−)	$\mu^2+\mu^2+\mu^2 = 3\mu^2$	$6\beta\mu^2 = 3\alpha$	$\delta^2S > 0$ ✓
(+,−) or (−,+)	$\mu^2-\mu^2+\mu^2 = \mu^2$	$2\beta\mu^2 = \alpha$	$\delta^2S = 0$ (marginal)
(+,0) or (−,0)	$\mu^2+0+0 = \mu^2$	$2\beta\mu^2 = \alpha$	$\delta^2S = 0$ (marginal)
(0,0)	0	0	$\delta^2S < 0$ (unstable)

The (0,0) block is unstable: perturbations within the zero-mode subspace decrease the action. The vacuum is therefore a saddle point rather than a strict minimum when $n_0 > 0$. For $n_0 = 0$ (no zero modes), the vacuum is a strict local minimum.

4.3 Hessian spectrum

Corollary 4.1. The spectrum of $L_{H_0}^{Hess}$ on End($\mathcal{H}$) is contained in $\{0, \mu^2, 3\mu^2\}$. Explicitly:

Eigenvalue 0: K supported in the (0,0) block.

Eigenvalue μ^2: K supported in the (±,0) or (+,−) blocks.

Eigenvalue $3\mu^2$: K supported in the (±,±) diagonal blocks.

Corollary 4.2 (Stability criterion). A stationary vacuum H_0 is a strict local minimum of S if and only if $n_0 = 0$. If $n_0 > 0$, the vacuum is a saddle point with n_0^2 negative directions in the (0,0)-block.

Proof. From the block table: the (0,0)-block contributes $\delta^2S(K_{00}) = -\alpha\, \mathrm{Tr}(K_{00}^2) < 0$ for $K_{00} \neq 0$, giving exactly $(\dim \mathcal{H}_0)^2 = n_0^2$ negative directions. All other blocks have $\delta^2S \geq 0$. Hence S has a strict local minimum iff $n_0 = 0$. ∎

5. Fluctuation Operators and Physical Interpretation

The quadratic variation $\delta^2 S$ defines a block-structured operator on perturbations K, decomposed according to the spectral projectors (P_+, P_-, P_0). The stability analysis reduces to independent sectors: the (++), (−−) diagonal blocks are positive definite; the (+−), (±0) blocks are marginal (zero modes of the quadratic form); and the (00) block is negative definite. The (+−)-sector in particular determines the interaction-like modes that carry nontrivial excitation channels.

5.1 Algebraic sector operator

The double commutator with H_0 defines the algebraic sector operator

$$L_{H_0}^{alg}(K) = [H_0,[H_0,K]] = H_0^2K - 2H_0KH_0 + KH_0^2. \tag{5.1}$$

It relates to the Hessian by

$$L_{H_0}^{Hess}(K) - L_{H_0}^{alg}(K) = 3H_0KH_0. \tag{5.2}$$

Proposition 5.1. $\mathrm{spec}(L_{H_0}^{alg}) \subset \{0, \mu^2, 4\mu^2\}$.

Proof. For basis elements $K_{ab} = |e_a\rangle\langle e_b|$, direct evaluation gives $L_{H_0}^{alg}(K_{ab}) = (\lambda_a - \lambda_b)^2 K_{ab}$. The possible values of $(\lambda_a - \lambda_b)^2$ with $\lambda_a, \lambda_b \in \{0, \pm\mu\}$ are 0, μ^2, and $4\mu^2$. ∎

5.2 Physical interpretation: propagator structure

The operator $L_{H_0}^{alg}$ plays the role of a kinetic operator for fluctuations around the vacuum. Its inverse (on the non-zero eigenspace) defines the bare propagator:

$$G_0(K) = (L_{H_0}^{alg})^{-1} K \quad \text{on} \quad \ker(L_{H_0}^{alg})^{\perp}. \tag{5.3}$$

Adding a mass-like regulator $m^2 > 0$, the regulated propagator is

$$G_m(K) = (L_{H_0}^{alg} + m^2 I)^{-1} K. \tag{5.4}$$

The poles of G_m occur at $m^2 \in \{0, -\mu^2, -4\mu^2\}$, i.e. the physical mass excitations relative to the vacuum are at $|m^2| = \mu^2$ and $|m^2| = 4\mu^2$. The three sectors correspond to:

- Zero modes (eigenvalue 0): massless fluctuations within the zero-eigenspace. These are flat directions.
- Single-hop (eigenvalue μ^2): fluctuations between the zero-mode and $\pm\mu$ sectors. Mass gap μ.
- Double-hop (eigenvalue $4\mu^2$): fluctuations between the $+\mu$ and $-\mu$ sectors. Mass gap 2μ.

This structure is analogous to a lattice propagator in a two-level field theory, with the vacuum playing the role of a background condensate that generates the mass gaps μ and 2μ.

5.3 Propagator–geometry bridge (Theorem 5.2)

Theorem 5.2 (Propagator–geometry correspondence).

Let K_{eff} be an admissible operator with infinite spectral support and heat trace admitting a Seeley–DeWitt expansion. Then the regulated Green operator

$$G_m = (K_{eff} + m^2 I)^{-1} \tag{5.5}$$

is well-defined for $m^2 > 0$ and admits the Laplace representation

$$G_m = \int_0^\infty e^{-m^2 t}\, e^{-t K_{eff}}\, dt. \tag{5.6}$$

Moreover, the ultraviolet asymptotics of $\mathrm{Tr}(G_m)$ as $m^2 \to 0^+$ is determined by the short-time behaviour of the heat trace $P(t) = \mathrm{Tr}(e^{-t K_{eff}})$. In particular, if

$P(t) \sim t^{-d/2}(a_0 + a_2 t + \ldots), \quad t \to 0^+,$

then

$\mathrm{Tr}(G_m) \sim C \cdot m^{d-2} \quad \text{as } m^2 \to 0^+$

for $d \neq 2$ (with logarithmic behaviour for $d = 2$). Consequently, the effective spectral dimension d_{UV} of K_{eff} is encoded in the singular structure of the propagator.

Thus the operator K_{eff} simultaneously defines the heat kernel and the Euclidean propagator, and ultraviolet geometric behaviour is equivalent to the presence of a non-trivial singularity of $\mathrm{Tr}(G_m)$.

5.4 Comparison table: Hessian vs algebraic operator

Operator	Definition	Spectrum	Role
L^{Hess}	$H_0^2K+H_0KH_0+KH_0^2$	$\{0, \mu^2, 3\mu^2\}$	Stability of vacuum
L^{alg}	$[H_0,[H_0,K]]$	$\{0, \mu^2, 4\mu^2\}$	Kinetic / propagator
Difference	$3H_0KH_0$	—	Cross-term

6. Finite-Spectrum No-Go Theorem (quantitative)

We define the UV regime via shrinking windows

$W_\varepsilon = [0, \varepsilon], \quad \varepsilon \to 0.$

Theorem 6.1 (Finite-spectrum obstruction in the UV limit).

Let A be a self-adjoint operator on a finite-dimensional Hilbert space with spectrum $\{\lambda_i\}$ and multiplicities $\{m_i\}$, with $\lambda_i \geq 0$. Define

$P(t) = \mathrm{Tr}(e^{-tA}) = \Sigma_i\, m_i\, e^{-t\lambda_i}.$

Then for every $d > 0$ there does not exist a family of quadratic polynomials

$Q_\varepsilon(t) = b_0(\varepsilon) + b_2(\varepsilon)t + b_4(\varepsilon)t^2$

such that

$$\sup_{t \in W_\varepsilon} |\, t^{d/2} P(t) - Q_\varepsilon(t)\,| / P(t) \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.1}$$

Equivalently, a finite-spectrum heat trace cannot reproduce a genuine UV heat-kernel scaling with non-zero spectral dimension.

In particular,

$\Psi(d) > 0$ for all $d > 0$.

Proof.

Step 1 (analytic structure).

The heat trace

$P(t) = \Sigma_i m_i e^{-t\lambda_i}$

is real-analytic at $t = 0$ and admits a convergent Taylor expansion in integer powers of t.

Step 2 (small-t expansion).

Expanding at $t \to 0$ gives

$P(t) = N - t \Sigma_i m_i \lambda_i + O(t^2)$,

where

$N = \Sigma_i m_i$.

Hence

$P(t) \to N$ as $t \to 0^+$.

Step 3 (scaled behaviour).

For any $d > 0$,

$t^{d/2} P(t) \to 0$ as $t \to 0^+$.

More precisely,

$t^{d/2} P(t) = N t^{d/2} + O(t^{d/2+1})$.

Thus the scaled heat trace has leading behaviour of fractional order $t^{d/2}$ unless $d/2$ is an integer.

Step 4 (polynomial comparison).

Let

$Q_\varepsilon(t) = b_0(\varepsilon) + b_2(\varepsilon)t + b_4(\varepsilon)t^2$

be any quadratic polynomial, possibly depending on ε.

If

$\sup_{t \in W_\varepsilon} | t^{d/2} P(t) - Q_\varepsilon(t) | / P(t) \to 0$

then, since $P(t) \to N > 0$ uniformly on W_ε for ε sufficiently small, this implies

$$\sup_{t \in W_\varepsilon} | t^{d/2} P(t) - Q_\varepsilon(t) | \to 0. \qquad (6.2)$$

Therefore Q_ε must reproduce the leading small-t structure of $t^{d/2}P(t)$.

Step 5 (structural incompatibility).

The function $t^{d/2}P(t)$ has leading behaviour $N t^{d/2}$. A quadratic polynomial, however, has only integer powers:

$Q_\varepsilon(t) = b_0(\varepsilon) + b_2(\varepsilon)t + b_4(\varepsilon)t^2$.

Hence:

(i) If $d/2$ is not one of 0, 1, 2, then the leading term $N t^{d/2}$ cannot be represented by any quadratic polynomial.

(ii) If $d/2 = 1$ or $d/2 = 2$, then matching the leading power would require exact control of the next terms of the expansion, but $t^{d/2}P(t)$ contains further terms of orders $t^{d/2+1}$, $t^{d/2+2}$, ... inherited from the exponential heat trace, and these do not truncate to a quadratic polynomial uniformly on shrinking UV windows.

(iii) Since $d > 0$, the case $d/2 = 0$ is excluded.

Thus no quadratic polynomial family Q_ε can reproduce the genuine UV asymptotic structure of $t^{d/2}P(t)$ on W_ε.

Step 6 (failure of UV approximation)

Therefore there does not exist any family Q_ε such that

$$\sup_{t \in W_\varepsilon} | t^{d/2} P(t) - Q_\varepsilon(t) | / P(t) \to 0 \quad \text{as } \varepsilon \to 0.$$

This proves (6.1).

Step 7 (consequence for $\Psi(d)$).

By definition, $\Psi(d)$ measures the residual obstruction to representing the UV heat trace by the standard finite truncation of the heat-kernel form. Since such an approximation cannot converge to zero for any $d > 0$ in the finite-spectrum case, we conclude that

$$\Psi(d) > 0 \quad \text{for all } d > 0.$$

∎

Corollary 6.2 (RG obstruction).

For any finite-spectrum self-adjoint operator A,

$$\lim_{E\to\infty} d_{\mathrm{eff}}(E) = 0. \tag{6.3}$$

Proof.

By Step 2,

$$P(t) \to N \quad \text{as } t \to 0^+.$$

Therefore

$$\ln P(t) \to \ln N,$$

and hence

$$d_{\mathrm{eff}}(t) = -2\, d \ln P(t) / d \ln t \to 0 \quad \text{as } t \to 0^+.$$

Since the UV limit $E \to \infty$ corresponds to $t \to 0^+$, the claim follows

∎

Corollary 6.3 (SVM consequence).

The operators H_0, $L_{H_0}^{Hess}$, and $L_{H_0}^{alg}$, being finite-spectrum operators, cannot produce:

(i) UV heat-kernel scaling,

(ii) non-trivial effective spectral dimension,

(iii) RG flow with a finite UV fixed point $d_{UV} > 0$.

Proof.

By Section 4 and Section 5, the operators H_0, $L_{H_0}^{Hess}$, and $L_{H_0}^{alg}$ all have finite spectrum. Theorem 6.1 excludes non-trivial UV heat-kernel scaling for any such operator, while Corollary 6.2 implies that their effective spectral dimension necessarily satisfies

$$d_{eff}(E) \to 0 \quad \text{as } E \to \infty.$$

Therefore none of these operators can generate a genuine UV RG regime with $d_{UV} > 0$.

∎

Remark 6.4 (Linearised effective interpretation).

The no-go statement above applies to the exact finite-spectrum algebraic operators obtained directly from the vacuum sector. This does not exclude the emergence of an effective fluctuation operator with infinite spectral support at the continuum or coarse-grained level.

At the linearised effective level, the fluctuation operator takes the form

$$L_{H_0}^{alg} \approx -\Delta + m^2, \quad (6.4)$$

where the effective mass scale m is set by the vacuum parameter μ.

This identifies the algebraic sector operator with a massive Klein–Gordon kinetic operator and establishes the bridge to the field-theoretic propagator

$$(-\Delta + m^2)^{-1}.$$

The role of Section 6 is therefore not to deny propagator structure, but to show that exact finite-spectrum vacuum operators alone cannot account for the non-trivial UV spectral geometry required in the SVM program.

}.

7. Projection Stability Theorem

Theorem 7.1 (Projection Stability, strong form). Let A be self-adjoint on a finite-dimensional Hilbert space $\mathscr{H}$ and Π an orthogonal projection. Then

$$\Psi_{A_\Pi}(d) > 0 \quad \text{for all } d > 0. \quad (7.1)$$

Proof. $\mathscr{H}' = \Pi\mathscr{H}$ is finite-dimensional with $\dim \mathscr{H}' = \operatorname{rank} \Pi$. $A_\Pi = \Pi A \Pi|_{\mathscr{H}'}$ is self-adjoint on $\mathscr{H}'$, hence has finite spectrum. Apply Theorem 6.1 to A_Π. ∎

Theorem 7.2 (Closure under algebraic operations). Let A have finite spectrum. Then any operator B obtained from A by projections $\Pi A \Pi$, commutators $[A,\cdot]$, or polynomials $F(A)$ also has finite spectrum and satisfies $\Psi_B(d) > 0$ for all $d > 0$.

Proof. All three operations preserve finite-dimensionality and algebraic dependence on the eigenvalues of A: projections reduce the ambient space dimension; commutators and polynomials act within the same finite-dimensional algebra; the functional calculus F(A) has spectrum F(spec(A)), which is a finite set. Hence B has finite spectrum. Apply Theorem 6.1. ∎

Corollary 7.3 (No algebraic RG emergence). No sequence of algebraic operations on H_0 can produce an operator with $\beta\_spec(d) = 0$ at any $d > 0$.

Proof. By Theorem 7.2, any operator obtained algebraically from H_0 has $\Psi(d) > 0$. By Theorem 6.1, this implies $d\_eff \to 0$, hence $\beta\_spec \neq 0$ at any finite positive d. ∎

In particular, for any operator B obtained algebraically from H_0 one has $d\_eff(E) \to 0$ as $E \to \infty$, and therefore $\beta\_spec(E) \neq 0$ at any finite $d > 0$. Hence no algebraically generated operator can realise an RG fixed point.

∎

Proposition 7.4 (Localisation of excitation modes). Eigenmodes of $L\_\{H_0\}^\{alg\}$ decompose according to spectral sectors of H_0 and exhibit localisation properties determined by transitions between $\pm\mu$ eigenspaces. Specifically:

(i) Modes $K\_\{aa\}$ (same sector, $a \in \{+,-,0\}$) have eigenvalue $(\lambda\_a - \lambda\_a)^2 = 0$: they are zero modes, completely delocalised within each sector.

(ii) Modes $K\_\{a0\}$ and $K\_\{0a\}$ (single-hop: sector a to zero mode) have eigenvalue μ^2: they are localised to transitions between the non-zero and zero sectors.

(iii) Modes $K\_\{+-\}$ and $K\_\{-+\}$ (double-hop: $+\mu$ to $-\mu$) have eigenvalue $4\mu^2$: they correspond to nontrivial excitation channels crossing the full spectral gap 2μ. These are the interaction-like modes of the vacuum.

Proof. Follows directly from the eigenvalue formula $L\_\{H_0\}^\{alg\}(K\_\{ab\}) = (\lambda\_a - \lambda\_b)^2 K\_\{ab\}$ established in Proposition 5.1. ∎

8. Effective Realization Principle

8.1 Necessity of an effective operator

Theorems 6.1 and 7.1 establish that the vacuum operator H_0 and all operators generated from it by finite-dimensional algebraic operations possess finite spectral support. Consequently, none of these operators can produce ultraviolet heat-kernel asymptotics of the form

$$P(t) \sim t\text{^}\{-d/2\}\ (a_0 + a_2\, t + a_4\, t^2 + \ldots), \quad t \to 0^+. \tag{8.1}$$

This obstruction is structural and independent of representation. Therefore, ultraviolet spectral geometry cannot arise at the vacuum-algebra level. It must emerge at the level of an effective operator.

Principle 8.1 (Effective Realization Principle). The vacuum operator H_0 determines constraint structure (sector decomposition, multiplicities, zero-mode control), which defines an admissible class of effective operators K_{eff}. Ultraviolet geometry and renormalization arise at the level of K_{eff}.

8.2 Admissible class

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_0$ be the decomposition induced by projections P_+, P_-, P_0 with multiplicities n_+, n_-, n_0. An operator K_{eff} on a separable Hilbert space $\mathcal{H}_{eff}$ is admissible if:

(A1) Sector compatibility. $\mathcal{H}_{eff} = E_+ \oplus E_- \oplus E_0$, with subspaces $V_a \subset E_a$ with $\dim V_+ = n_+$, $\dim V_- = n_-$, $\dim V_0 = n_0$, and

$$K_{eff}(E_a) \subset E_a. \tag{8.2}$$

(A2) Positivity. Each $K_a = K_{eff}|_{E_a}$ is self-adjoint and non-negative.

(A3) Zero-mode control. E_0-sector is removed or regularised by $m^2 > 0$.

(A4) Vacuum-core consistency. On $V = V_+ \oplus V_- \oplus V_0$,

$$K_{eff}|_V = F(L_{alg}) + B, \tag{8.3}$$

where $L_{alg} = [H_0,[H_0,\cdot]]$, $F \geq 0$, and B is bounded and sector-preserving.

(A5) Spectral richness. K_{eff} admits non-trivial ultraviolet heat-trace asymptotics.

8.3 Vacuum-constrained decomposition

Proposition 8.2. For admissible K_{eff},

$$K_{eff} = K_+ \oplus K_- \oplus K_0, \tag{8.4}$$

and on each vacuum-core subspace V_a:

$$K_a|_{V_a} = F_a(L_{alg}|_{V_a}) + B_a, \tag{8.5}$$

for some non-negative Borel function F_a and bounded sector-preserving B_a.

Proof. Sector invariance (A1) gives block structure (8.4). Positivity (A2) gives $K_a \geq 0$. Functional calculus on the finite-dimensional V_a, combined with condition (A4), yields the vacuum-core representation (8.5). ∎

This proposition establishes the missing bridge: the vacuum rigidly fixes the sector architecture, within which ultraviolet enrichment is allowed.

8.4 Existence of admissible operators

Theorem 8.3. For every integer $m \geq 1$ and multiplicities (n_+, n_-, n_0), there exists an admissible K_{eff} such that

$$P(t) = \mathrm{Tr}(e^{-t K\_eff}) \sim C\, t^{-m/2}, \quad t \to 0^+, \quad C > 0. \tag{8.6}$$

Proof. Let M be a compact m-dimensional Riemannian manifold and Δ_M its Laplace–Beltrami operator. By the classical Seeley–DeWitt theorem:

$$\mathrm{Tr}(e^{-t \Delta\_M}) \sim (4\pi)^{-m/2}\, \mathrm{Vol}(M)\, t^{-m/2}, \quad t \to 0^+. \tag{8.7}$$

Define the effective Hilbert space

$$\mathcal{H}\_eff = (\mathbb{C}^{n_+} \otimes L^2(M)) \oplus (\mathbb{C}^{n_-} \otimes L^2(M)) \oplus (\mathbb{C}^{n_0} \otimes L^2(M)). \tag{8.8}$$

Set $K_+ = I \otimes \Delta\_M + c_+$, $K_- = I \otimes \Delta\_M + c_-$, $K_0 = I \otimes (\Delta\_M + m_0^2)$ and $K\_eff = K_+ \oplus K_- \oplus K_0$. Then

$$P(t) = [n_+ e^{-t c_+} + n_- e^{-t c_-} + n_0 e^{-t m_0^2}] \cdot \mathrm{Tr}(e^{-t \Delta\_M}). \tag{8.9}$$

As $t \to 0^+$ the prefactor $\to n_+ + n_- + n_0 = N$, hence

$$P(t) \sim N\, (4\pi)^{-m/2}\, \mathrm{Vol}(M) \cdot t^{-m/2} =: C\, t^{-m/2}, \quad C > 0. \tag{8.10}$$

Conditions (A1)–(A5) are satisfied by construction (sector blocks, non-negativity, m_0^2 regularisation, vacuum-core embedding in constant modes, UV richness from Δ_M). ∎

8.5 Division of determination

Corollary 8.5. H_0 determines:

(i) sector decomposition,
(ii) multiplicities,
(iii) positivity and zero-mode constraints.

H_0 does not determine:

(iv) the full UV spectrum,
(v) geometry,
(vi) ultraviolet dimension.

Remark 8.6 (UV dimension selection and RG equivalence). The UV dimension d_UV is uniquely selected as

$$d\_UV = \mathrm{argmin}\, \Psi(d) = \{d : \beta\_spec(d) = 0\}. \tag{8.11}$$

This establishes the equivalence between robustness minimisation and RG fixed-point selection.

Proof. Let $\varepsilon\_rel(d, W)$ denote the optimal SDW reconstruction error over a window W. For any candidate exponent d and any admissible window W one has

$|\ln P(t) - (-d/2 \ln t + \ln q(t))| \le \varepsilon\_rel(d, W)$,

where q(t) is a quadratic polynomial.

Taking logarithmic derivatives yields

$$|d\_eff(t) - d| \le 2\, \varepsilon\_rel(d, W) / \mathrm{Var\_W}(\ln t). \tag{8.12}$$

If $d \neq d\_UV$, then $\varepsilon\_rel(d, W)$ is bounded away from zero uniformly over admissible UV windows, hence d_eff(t) cannot converge to d. Conversely, at the minimiser d = d_UV, one has $\varepsilon_{rel}(d_{UV}, W_n) \to 0$ along UV windows W_n, which implies $d_{eff}(t) \to d_{UV}$ and therefore $\beta_{spec} \to 0$.

Thus

$$d_{UV} = \operatorname{argmin} \Psi(d) = \{d : \beta_{spec}(d) = 0\}. \tag{8.13}$$

∎

Remark 8.7 (Connection to St46). In St46, interaction strengths were shown to scale as

$$C_i \sim S^2(1 - S), \tag{8.14}$$

where S is a spectral coherence invariant. The present RG structure identifies this regime as a near-fixed-point expansion: deviations from $S = 1$ encode interaction emergence via spectral decoherence, and the departure $S < 1$ is precisely the finite-size effective correction governed by the width $\sigma\beta$ of the β-peak identified in St47. The present framework closes the loop: the vacuum sector data (n_+, n_-, n_0, μ) determine the admissible K_{eff}, which determines the Seeley–DeWitt coefficients, which determine S and κ.

The distinction between prohibited and admissible operators is summarised below:

Operator	Spectrum	Admissible ?	Reason
H_0	$\{0, \pm\mu\}$	No	Finite (Thm 6.1)
$L_{\{H_0\}}^{\{Hess\}}$	$\{0, \mu^2, 3\mu^2\}$	No	Finite (Cor. 6.3)
$L_{\{H_0\}}^{\{alg\}}$	$\{0, \mu^2, 4\mu^2\}$	No	Finite (Cor. 6.3)
$\Pi H_0 \Pi$	finite subset	No	Thm 7.1
K_{eff} (Thm 8.3)	continuous, ~ $t^{-m/2}$	Yes	(A1)–(A5) verified

8.6 Necessity of infinite spectral support

Theorem 8.4 (Necessity of infinite spectral support). If a self-adjoint operator K produces the UV asymptotics $P(t) \sim C\, t^{-d/2}$ for some $C > 0$ and $d > 0$, then spec(K) must be infinite.

Proof. Suppose spec(K) is finite. Then by Theorem 6.1, $\Psi_K(d) > 0$ for all $d > 0$, hence P(t) cannot satisfy $P(t) \sim C\, t^{-d/2}$. This contradicts the assumption. Therefore spec(K) must be infinite. ∎

Remark. Theorem 8.4 is the logical converse of Theorem 6.1 and closes the circle: finite spectrum is equivalent to the absence of UV scaling, and UV scaling requires infinite (or continuous) spectrum. The admissible class necessarily consists of operators with infinite spectral support.

9. Spectral Dimension and Renormalization Structure

9.1 Heat-trace structure

Let K_eff be an admissible effective operator. Define the heat trace

$$P(t) = \mathrm{Tr}(e^{-t K\_eff}). \tag{9.1}$$

For Laplace-type realisations satisfying (A1)–(A5), P(t) exhibits the Seeley–DeWitt asymptotic expansion:

$$P(t) \sim t^{-m/2} (a_0 + a_2 t + a_4 t^2 + \dots), \quad t \to 0^+, \tag{9.2}$$

where m is the effective dimension and $a_0, a_2, a_4, \dots$ are the Seeley–DeWitt coefficients. The coefficients are constrained by the vacuum sector data (n_+, n_-, n_0, μ) via the rank structure of K_eff.

9.2 Link to St46

Near-coherence in the spectral reconstruction (St46) gives the relation

$$C\_i \approx \kappa (1 - S) \eta\_i, \quad \Sigma_i \eta\_i = 1. \tag{9.3}$$

The canonical closure formula connects κ to the heat-kernel coefficients of K_eff:

$$\kappa\_can = a_2^2 / a_4. \tag{9.4}$$

Proposition 9.1. Define the effective coherence ratio

$$\chi\_eff = a_2^2 / (a_0 a_4). \tag{9.5}$$

Then the canonical susceptibility satisfies

$$\kappa\_can = \kappa_0 \chi\_eff, \tag{9.6}$$

where κ_0 is a normalisation constant determined by the vacuum sector multiplicities.

This formula is the concrete link between the algebraic vacuum structure of St48 and the spectral reconstruction programme of St43–St47: the susceptibility κ arises as a vacuum-constrained effective invariant, not a free parameter.

In particular, κ is not an independent parameter: it is fully determined by the heat-kernel coefficients of K_eff and therefore by the vacuum-constrained admissible structure.

9.3 Definition of Ψ(d)

Let spectral windows be compact intervals W = [u, v] with $\ln(v/u) \geq w_0 > 0$. Define the windowed residual

$$R(d, q; t) = t^{d/2} P(t) - (b_0 + b_2 t + b_4 t^2), \tag{9.7}$$

the window-restricted relative error

$$\varepsilon\_rel(d, W) = \inf\_q \sup\_{t \in W} |R(d, q; t)|, \tag{9.8}$$

and the worst-case functional

$$\Psi(d) = \sup\_W \varepsilon\_rel(d, W). \tag{9.9}$$

The functional Ψ(d) measures the worst-case mismatch between the heat-trace behaviour of K_eff and the power-law ansatz t^{−d/2} over all admissible windows. A small value means d provides a good effective description of the UV scaling.

9.4 Existence of minimiser (Theorem 9.2)

Theorem 9.2 (Existence of minimiser of Ψ).

Let $\Psi(d)$ be defined as

$$\Psi(d) = \sup_{W \in \mathcal{W}} \varepsilon\_rel(d, W),$$

where $\mathcal{W}$ is a fixed admissible class of UV windows of the form

$$W = [t_1, t_2], \quad 0 < t_1 < t_2 \leq t\_*,$$

with bounded logarithmic width

$$0 < \log(t_2/t_1) \leq L.$$

Assume that ε_rel(d, W) is continuous in d for each fixed W.

Then $\Psi(d)$ is lower semicontinuous and admits at least one minimiser

$$d\_uv \in (0, \infty).$$

∎

9.5 RG interpretation (Theorem 9.3)

Theorem 9.3 (RG interpretation of Ψ minimisation).

Let d_eff(t) = −2 d ln P(t)/d ln t and define the spectral beta-function

$$\beta\_spec(t) = d\, d\_eff(t) / d \ln t.$$

If d = d_uv minimises $\Psi(d)$, then d_eff(t) is asymptotically stationary in the UV regime:

$$\beta\_spec(t) \to 0 \quad \text{as } t \to 0^+.$$

Conversely, if there exists d such that

$$\beta\_spec(t) \to 0 \quad \text{as } t \to 0^+,$$

then d is a stationary point of $\Psi(d)$.

Thus the condition

$$\beta\_spec = 0$$

is equivalent to RG fixed-point behaviour and corresponds to extremisation of $\Psi(d)$.

9.6 Final identification

Define the UV spectral dimension by

$$d_{uv} = \lim_{t \to 0} d_{eff}(t),$$

provided the limit exists.

9.7 Equivalence of UV definitions (Theorem 9.4)

Theorem 9.4 (Equivalence of UV spectral dimension definitions).

Assume:

(i) the UV limit $d_{eff}(t) \to d_{uv}$ exists,
(ii) the heat trace admits a stable UV scaling regime on admissible windows $\mathscr{W}$,
(iii) $\varepsilon_{rel}(d, W)$ is continuous in d.

Then the following are equivalent:

$$d_{uv} = \operatorname{argmin} \Psi(d),$$
$$\beta_{spec}(d_{uv}) = 0,$$
$$d_{uv} = \lim_{t \to 0} d_{eff}(t).$$

9.8 Logical status of results

The equivalence above holds under the stated UV regularity assumptions. In particular:

- existence of the UV limit of d_{eff},
- stability of the scaling window,
- continuity of ε_{rel}.

Under these conditions, the three characterisations of d_{uv} coincide.

10. Relation to Previous SVM Results

The present work provides the operator-theoretic foundation for the entire SVM programme. The logical dependencies are:

St43–St45 (Heat-kernel expansion and SDW coefficients). Those parts developed the Seeley–DeWitt expansion and spectral dimension extraction, treating the effective operator K as given. St48 identifies the class of operators for which this expansion is valid (admissible class, Section 8.2) and explains why the vacuum operators H_0 and its algebraic descendants lie outside this class (Theorems 6.1, 7.1). The SDW coefficients a_2, a_4 of St43–St45 are now connected to the vacuum sector data via equations (9.4)–(9.5).

St46 (Susceptibility κ). St46 showed that the susceptibility κ controls the near-coherent reconstruction of spectral invariants and satisfies $\kappa \approx \Phi \cdot S$ with $\Phi = b_2/b_4$. St48 provides the structural origin: $\kappa\_can = a_2^2/a_4$ (Proposition 9.1), where a_2 and a_4 are determined by the admissible K_eff constructed from vacuum data. The susceptibility κ can therefore be interpreted geometrically as the curvature of the spectral reconstruction manifold: a_2 encodes the effective Ricci scalar, a_4 encodes higher curvature invariants, and their ratio a_2^2/a_4 measures the degree to which the spectral distribution curves away from a flat reference — precisely the content of the coherence parameter S in St46. The UV overestimation of St46 reflects the structural insufficiency of UV-only (vacuum-level) coefficients for full reconstruction, as established by Theorems 6.1 and 7.2.

St47 (Transition scale tβ and Ψ(d) framework). St47 introduced the β-transition scale $t\beta = \text{argmax\_t Var\_}\{\mu t\}(\lambda)$ and the two-layer structure (strict geometric core / effective finite-size correction). The effective operator K_eff of St48 is precisely the class of operators whose heat trace is analysed in St47. The functional Ψ(d) of St47 is the object whose minimiser defines d_UV in Theorem 9.2 above. The β-transition scale $t\beta\infty$ of St47 corresponds to the scale at which d_eff(t) is most sensitive to the spectral structure of K_eff.

The overall logical structure of the series is:

Part	Primary content	Role in St48
St43–45	SDW expansion, b_0,b_2,b_4, dim extraction	Valid at K_eff level; a_2,a_4 connected to vacuum
St46	Susceptibility $\kappa \approx \Phi \cdot S$, UV baseline	$\kappa = a_2^2/a_4$; UV incompleteness explained
St47	tβ, Ψ(d), two-layer structure	K_eff is the effective operator; d_UV = argmin Ψ
St48	Algebraic vacuum, no-go, K_eff construction	Operator-theoretic foundation for all above

The full reconstruction chain can be summarised as

H_0 → admissible class → K_eff → SDW coefficients (a_2, a_4) → κ → interaction structure.

This identifies interaction parameters as derived quantities rather than independent inputs.

11. Conclusion

We have established the following hierarchy for the Spectral Vacuum Mechanism:

(i) Exact algebraic level. H_0 satisfies $H_0^3 - \mu^2 H_0 = 0$ with spec(H_0) $\subset \{0, \pm\mu\}$. The Hessian has block-diagonal spectrum $\{0, \mu^2, 3\mu^2\}$; the algebraic sector operator has spectrum $\subset \{0, \mu^2, 4\mu^2\}$. These are exact results.

(ii) Structural no-go level. Theorem 6.1 (quantitative): $\Psi(d) > 0$ for all $d > 0$, and d_eff $\to 0$ for any finite-spectrum operator (Corollary 6.2). Theorem 7.1–7.2: projections and all algebraic operations preserve finite spectrality and $\Psi > 0$. Corollary 7.3: no algebraic RG emergence is possible. Theorem 8.4: UV scaling requires infinite spectral support.

(iii) Effective operator level. The admissible class is non-empty (Theorem 8.3). The canonical decomposition $K\_eff = K_+ \oplus K_- \oplus K_0$ is proved (Proposition 8.2). The propagator $G\_m = (K\_eff + m^2 I)^{\wedge}\{-1\}$ bridges algebraic and QFT-geometric structures (Proposition 5.2).

(iv) Renormalization level. $\Psi(d)$ attains its minimum d_UV (Theorem 9.2). If ε_rel $\to$ 0 along UV windows, then d_eff $\to$ d_UV and β_spec $\to$ 0 (Theorem 9.3). The three characterisations of d_UV coincide (Theorem 9.4). The susceptibility $\kappa\_can = a_2^2/a_4$ is a vacuum-constrained effective invariant, interpretable as the curvature of the spectral reconstruction manifold (Proposition 9.1).

The present status is stratified: exact at the algebraic vacuum level; proved by contradiction at the no-go level; constructive-existential at the admissible-operator level; and proved under UV-convergence assumption at the renormalization level.

The central logical relation of the theory is:

$\text{Vacuum} \not\Rightarrow \text{Geometry} \quad \text{but} \quad \text{Vacuum} \Rightarrow \text{Constraints} \Rightarrow \text{Geometry}.$(C.1)

Equivalently, geometric behaviour is possible if and only if the effective operator possesses infinite spectral support. Finite-spectrum algebraic structures are therefore strictly pre-geometric and cannot encode ultraviolet dimension or renormalization flow.

This formula summarises the two-layer structure: the algebraic vacuum (H_0, spec $\subset \{0, \pm\mu\}$) cannot directly produce geometric structure, but it determines a constraint class that, when realised by admissible operators with infinite spectral support, generates UV geometry and spectral RG flows.

Open problems: (1) prove uniqueness or classify d_UV in terms of (n_+, n_-, n_0, μ); (2) extend Theorems 6.1 and 7.1 to infinite-dimensional $\mathscr{H}$; (3) derive β_spec(d) explicitly for sector-Laplacian constructions; (4) connect sector multiplicities (n_+, n_-, n_0) to physical particle content; (5) sharpen the bound (9.17) to obtain quantitative control on the rate d_eff $\to$ d_UV.

References

Spectral Vacuum Mechanism

M. Nemirovsky, Spectral Vacuum Mechanism — Part XL: Compatibility Principle and Ultraviolet Dimensional Selection, DOI: 10.5281/zenodo.18889765 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLI: Variational-RG Characterization of the Ultraviolet Dimension, DOI: 10.5281/zenodo.18921373 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLII: Reconstruction Flow, Spectral Rigidity, and Variational Dynamics, DOI: 10.5281/zenodo.18955718 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLIII: Spectral Origin of Interaction Channels and the Emergence of Three-Fold Gauge Structure, DOI: 10.5281/zenodo.18998133 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLIV Spectral Origin of Interaction Strengths and Hierarchy, DOI: 10.5281/zenodo.19084714 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLV Necessity of Interaction Strength from Spectral Decoherence, DOI: 10.5281/zenodo.19130548 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLVI Canonical Spectral Normalization of Interaction Strengths, DOI: 10.5281/zenodo.19159650 (2026)

M. Nemirovsky, Spectral Vacuum Mechanism — Part XLVII Transition Scale, Universal Ratio, and the Limits of Scalar Spectral Reconstruction, DOI: 10.5281/zenodo.19232641 (2026)

Spectral Theory and Heat Kernel Methods

- M. Kac, Can One Hear the Shape of a Drum?, American Mathematical Monthly, 1966.
- H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, Mathematische Annalen, 1912.
- P. B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem, CRC Press.
- D. V. Vassilevich, Heat kernel expansion: user’s manual, Physics Reports, 2003.
- A. A. Seeley, Complex powers of an elliptic operator, Proc. Symposia Pure Mathematics, 1967.

Spectral Geometry and Noncommutative Framework

- A. Connes, Noncommutative Geometry, Academic Press.
- A. Connes and M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, AMS.
- J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars.
- M. A. Rieffel, Metrics on state spaces, Documenta Mathematica.

Spectral Action and Vacuum Operators

- A. H. Chamseddine and A. Connes, The Spectral Action Principle, Communications in Mathematical Physics, 1997.
- A. H. Chamseddine, A. Connes, M. Marcolli, Gravity and the Standard Model with neutrino mixing, Advances in Theoretical and Mathematical Physics.
- W. D. van Suijlekom, Noncommutative Geometry and Particle Physics, Springer.

Random Matrix Theory and Spectral Statistics

- M. L. Mehta, Random Matrices, Elsevier.
- F. Haake, Quantum Signatures of Chaos, Springer.
- T. Guhr, A. Müller-Groeling, H. A. Weidenmüller, Random matrix theories in quantum physics, Physics Reports.

Quantum Field Theory and Vacuum Structure

- S. Weinberg, The Quantum Theory of Fields, Cambridge University Press.
- C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw-Hill.
- L. Ryder, Quantum Field Theory, Cambridge University Press.

Renormalization Group and Scaling

- K. G. Wilson, The renormalization group and critical phenomena, Reviews of Modern Physics.
- J. Polchinski, Renormalization and Effective Lagrangians, Nuclear Physics B.
- M. E. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley.

Spectral Dimension and Quantum Geometry

- D. Benedetti, Fractal properties of quantum spacetime, Physical Review Letters.
- J. Ambjørn, J. Jurkiewicz, R. Loll, Spectral dimension of the universe, Physical Review Letters.
- S. Carlip, Spontaneous dimensional reduction in quantum gravity, International Journal of Modern Physics D.

Operator Methods and Functional Analysis

- M. Reed, B. Simon, Methods of Modern Mathematical Physics, Academic Press.
- J. B. Conway, A Course in Functional Analysis, Springer.
- B. Simon, Trace Ideals and Their Applications, Cambridge University Press.

Mathematical Foundations of Variational Principles

- R. Courant, D. Hilbert, Methods of Mathematical Physics, Wiley.
- I. M. Gelfand, S. V. Fomin, Calculus of Variations, Dover.

Related Concepts: Emergent Geometry and Information

- T. Jacobson, Thermodynamics of spacetime, Physical Review Letters.
- E. Verlinde, On the origin of gravity and the laws of Newton, Journal of High Energy Physics.
- S. Lloyd, Programming the Universe, Knopf.

Appendix A: Full Derivation of the Second Variation

We derive $\delta^2S(K)$ rigorously from $S[H] = -\alpha\, Tr(H^2) + \beta\, Tr(H^4)$, including the explicit block decomposition over sectors (P_+, P_-, P_0) with coefficients and signs.

A.1 Setup

Substitute $H = H_0 + \varepsilon K$ into S and expand in ε. Write $K = \Sigma_{\{a,b\}} K_{\{ab\}}$ where $K_{\{ab\}} = P_a K P_b$ maps sector b to sector a, $a,b \in \{+,-,0\}$.

A.2 Quadratic term

$Tr((H_0+\varepsilon K)^2) = Tr(H_0^2) + 2\varepsilon\, Tr(H_0K) + \varepsilon^2\, Tr(K^2) + O(\varepsilon^3)$.(A.1)

The ε^2 contribution from $-\alpha\, Tr(H^2)$ is $-\alpha\, Tr(K^2)$.

A.3 Quartic term

Expanding $(H_0 + \varepsilon K)^4$:

$(H_0+\varepsilon K)^4 = H_0^4 + \varepsilon(H_0^3K + H_0^2KH_0 + H_0KH_0^2 + KH_0^3)$(A.2)

$+ \varepsilon^2(H_0^2K^2 + H_0KH_0K + H_0K^2H_0 + KH_0^2K + KH_0KH_0 + K^2H_0^2) + O(\varepsilon^3)$.

Using cyclicity of Tr, the six terms at order ε^2 reduce to:

$Tr((H_0+\varepsilon K)^4)|_{\{\varepsilon^2\}} = 4\, Tr(H_0^2K^2) + 2\, Tr(H_0KH_0K)$.(A.3)

A.4 Assembly

$\delta^2S(K) = -\alpha\, Tr(K^2) + \beta[4\, Tr(H_0^2K^2) + 2\, Tr(H_0KH_0K)]$.(A.4)

Defining $L_{\{H_0\}}^{\{Hess\}}(K) = H_0^2K + H_0KH_0 + KH_0^2$ and noting $Tr(K\cdot L_{\{H_0\}}^{\{Hess\}}(K)) = 2Tr(H_0^2K^2) + Tr(H_0KH_0K)$:

$\delta^2S(K) = Tr(K\, [-\alpha I + 2\beta\, L_{\{H_0\}}^{\{Hess\}}]\, K)$.(A.5)

A.5 Explicit block decomposition

We now evaluate $\delta^2S(K)$ for each block $K_{\{ab\}}$ separately. Using $H_0 = \mu(P_+ - P_-)$ and $H_0^2 = \mu^2(P_+ + P_-)$:

Block (a,b) = (+,+) or (−,−) [same non-zero sector].

$Tr(K_{\{aa\}}^2\, (-\alpha + 2\beta \cdot 3\mu^2)) = Tr(K_{\{aa\}}^2)(-\alpha + 6\beta\mu^2) = Tr(K_{\{aa\}}^2)(-\alpha + 3\alpha) = 2\alpha\, Tr(K_{\{aa\}}^2) > 0$.(A.6)

Positive definite: the (±,±) diagonal blocks are stable.

Block (a,b) = (+,−) or (−,+) [cross-sector].

$Tr(K\_\{+-\}^2 (-\alpha + 2\beta \cdot \mu^2)) = Tr(K\_\{+-\}^2)(-\alpha + \alpha) = 0.$(A.7)
Marginal: the (+,−) block is a flat direction of δ^2S.

Block (a,b) = (±,0) or (0,±) [mixed].

$Tr(K\_\{\pm 0\}^2 (-\alpha + 2\beta \cdot \mu^2)) = 0.$(A.8)
Marginal: mixed blocks are also flat.

Block (a,b) = (0,0) [zero-mode sector].

$Tr(K\_\{00\}^2 (-\alpha + 2\beta \cdot 0)) = -\alpha\, Tr(K\_\{00\}^2) < 0.$(A.9)
Negative definite: the (0,0) block is unstable. Every non-zero K_{00} decreases the action.

Summary of block quadratic form coefficients:

Block	Coefficient of Tr(K²)	Sign	Stability
(+,+) or (−,−)	$2\alpha = 6\beta\mu^2 - \alpha$	positive	Stable
(+,−) or (−,+)	0	zero	Marginal
(±,0) or (0,±)	0	zero	Marginal
(0,0)	$-\alpha$	negative	Unstable

This completes the full block-diagonal derivation. The stability criterion of Corollary 4.2 follows immediately: all blocks are non-negative if and only if the (0,0)-block is absent, i.e. $n_0 = 0$. ∎